

# Asymptotic formulas for general colored partition functions

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## Abstract

In 1917, Hardy and Ramanujan obtained the asymptotic formula for the classical partition function  $p(n)$ . The classical partition function  $p(n)$  has been extensively studied. Recently, Luca and Ralaivaosaona obtained the asymptotic formula for the square-root function. Many mathematicians have paid much attention to congruences on some special colored partition functions. In this paper, we investigate the general colored partition functions. Given positive integers  $1 = s_1 < s_2 < \cdots < s_k$  and  $\ell_1, \ell_2, \dots, \ell_k$ . Let  $g(\mathbf{s}, \mathbf{l}, n)$  be the number of  $\ell$ -colored partitions of  $n$  with  $\ell_i$  of the colors appearing only in multiplies of  $s_i$  ( $1 \leq i \leq k$ ), where  $\ell = \ell_1 + \cdots + \ell_k$ . By using the elementary method we obtain an asymptotic formula for the partition function  $g(\mathbf{s}, \mathbf{l}, n)$  with an explicit error term.

**Keyword:** colored partition; partition function; asymptotic formula; Gaussian integral

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# 1 Introduction

Let  $p(n)$  denote the number of partitions of  $n$ , i.e.

$$n = a_1 + \cdots + a_k$$

with integers  $1 \leq a_1 \leq \cdots \leq a_k$ . The generating function of  $p(n)$  is

$$f(z) = 1 + \sum_{n=1}^{\infty} p(n)z^n = \prod_{n=1}^{\infty} \frac{1}{1 - z^n}. \quad (1.1)$$

Ramanujan [16] obtained many congruent identity for  $p(n)$ . Hardy and Ramanujan [9] and Uspensky [18] independently proved that

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right). \quad (1.2)$$

An elementary proof for this formula is given by Erdős [8] with no explicit constant  $(4\sqrt{3})^{-1}$ . Lehmer [12] gave the series for the partition function  $p(n)$ . Odlyzko [15, (1.6)] gave an asymptotic formula for  $p(n)$  with an explicit error term. That is,

$$p(n) = \frac{1 + O(n^{-1/2})}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right). \quad (1.3)$$

The partition function  $p(n)$  has a long history and generates varieties.

Recently, Luca and Ralaivaosaona [14] obtained the asymptotic formula for the square-root function  $q(n)$ , which is defined to be the number of solutions of

$$n = [\sqrt{a_1}] + [\sqrt{a_2}] + \cdots + [\sqrt{a_k}]$$

with integers  $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k$ . For related results, one may refer to Balasubramanian and Luca [2] and Chen and Li [7] and [13].

Let  $p_k(n)$  be the number of 2-color partitions of  $n$  where one of the colors appears only in parts that are multiples of  $k$ . The generating function of  $p_k(n)$  is

$$1 + \sum_{n=1}^{\infty} p_k(n)z^n = \prod_{n=1}^{\infty} \frac{1}{(1 - z^n)(1 - z^{kn})}.$$

Chan [3], Kim [11] and Sinick [17] studied some results of the case  $k = 2$ . Recently, Ahmed, Baruah and Dastidar [1] and Chern [6] obtained many congruences of  $p_k(n)$  for some  $k$ .

By further analogy, Chan and Cooper [4] and Chen [5] considered a special partition function  $c(n)$  which is the number of 4-colored partitions of  $n$  with two of the colors appearing only in multiplies of 3. The generating function of  $c(n)$  is

$$1 + \sum_{n=1}^{\infty} c(n)z^n = \prod_{n=1}^{\infty} \frac{1}{(1-z^n)^2(1-z^{3n})^2}.$$

In this paper, we focus on the asymptotic formula for the general colored partition functions.

Given integers  $1 = s_1 < s_2 < \cdots < s_k$  and  $\ell_1, \ell_2, \dots, \ell_k$ . Let  $g(\mathbf{s}, \mathbf{l}, n)$  be the number of  $\ell$ -colored partitions of  $n$  with  $\ell_i$  of the colors appearing only in multiplies of  $s_i$  ( $1 \leq i \leq k$ ), where  $\ell = \ell_1 + \cdots + \ell_k$ . Write

$$\mathbf{s} = (s_1, s_2, \dots, s_k)$$

and

$$\mathbf{l} = (\ell_1, \ell_2, \dots, \ell_k).$$

We call  $g(\mathbf{s}, \mathbf{l}, n)$  the  $(\mathbf{s}, \mathbf{l})$ -colored partition function. For convenience, we define  $g(\mathbf{s}, \mathbf{l}, 0) = 1$  and  $g(\mathbf{s}, \mathbf{l}, n) = 0$  for all  $n < 0$ . The generating function of  $g(\mathbf{s}, \mathbf{l}, n)$  is

$$\sum_{n=0}^{\infty} g(\mathbf{s}, \mathbf{l}, n)x^n = \prod_{n=1}^{\infty} \frac{1}{(1-z^{s_1 n})^{\ell_1} \cdots (1-z^{s_k n})^{\ell_k}}. \quad (1.4)$$

In this paper, the following result is proved.

**Theorem 1.1.** *For any given  $\varepsilon > 0$  (small), we have*

$$g(\mathbf{s}, \mathbf{l}, n) = c(\mathbf{s}, \mathbf{l})n^{d(\mathbf{l})} \exp \left( \pi \sqrt{\frac{2a(\mathbf{s}, \mathbf{l})n}{3}} \right) + O \left( n^{d(\mathbf{l}) - \frac{1}{4} + \varepsilon} \exp \left( \pi \sqrt{\frac{2a(\mathbf{s}, \mathbf{l})n}{3}} \right) \right),$$

where

$$a(\mathbf{s}, \mathbf{l}) = \sum_{i=1}^k \frac{\ell_i}{s_i}, \quad d(\mathbf{l}) = -\frac{3}{4} - \frac{1}{4}(\ell_1 + \cdots + \ell_k),$$

$$c(\mathbf{s}, \mathbf{l}) = 2^{-(3\ell_1 + \cdots + 3\ell_k + 5)/4} 3^{-(\ell_1 + \cdots + \ell_k + 1)/4} a(\mathbf{s}, \mathbf{l})^{(\ell_1 + \cdots + \ell_k + 1)/4} s_1^{\ell_1/2} \cdots s_k^{\ell_k/2}.$$

**Remark 1.2.** *By employing the Tauberian theorem of Ingham [10], under its form in the special case, it is possible to get an asymptotic formula for  $g(\mathbf{s}, \mathbf{l}, n)$  without the error term. We do not intend to give the details here.*

Let  $k = 2, s_1 = 1, s_2 = 3$  and  $\ell_1 = \ell_2 = 2$ . By Theorem 1.1, we can give an asymptotic formula for  $c(n)$ , that is,

$$c(n) = \frac{1}{3\sqrt{6}n^{7/4}} \exp\left(\frac{4}{3}\pi\sqrt{n}\right) + O\left(n^{-2+\varepsilon} \exp\left(\frac{4}{3}\pi\sqrt{n}\right)\right).$$

We believe that there are many congruences for the  $(\mathbf{s}, \mathbf{l})$ -colored partition functions as many known various partition functions.

## 2 The main ingredients

For convenience, let

$$a = a(\mathbf{s}, \mathbf{l}) = \sum_{i=1}^k \frac{\ell_i}{s_i}, \quad g(n) = g(\mathbf{s}, \mathbf{l}, n), \quad c_1 = \pi\sqrt{\frac{2}{3}}, \quad c_2 = \frac{1}{4\sqrt{3}}. \quad (2.1)$$

Then (1.3) becomes

$$p(n) = \frac{c_2 + O(n^{-1/2})}{n} \exp(c_1\sqrt{n}). \quad (2.2)$$

If  $k = 1$  and  $\ell_1 = 1$ , then  $g(n) = p(n)$ . In this case, Theorem 1.1 follows from (2.2). Now we assume that  $k + \ell_1 \geq 3$ . So  $as_k > 1$ .

**Lemma 2.1.** *We have*

$$g(n) = \sum_{\mathbf{u} \in U_n} \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i}} p(u_{i,j}),$$

where  $U_n$  is the set of all  $\ell_1 + \cdots + \ell_k$  tuples

$$\mathbf{u} = (u_{1,1}, \dots, u_{1,\ell_1}, \dots, u_{k,1}, \dots, u_{k,\ell_k})$$

of nonnegative integers with

$$s_1 u_{1,1} + \cdots + s_1 u_{1,\ell_1} + \cdots + s_k u_{k,1} + \cdots + s_k u_{k,\ell_k} = n.$$

*Proof.* By (1.1) and (1.4), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} g(n) z^n &= \left( \sum_{n=0}^{\infty} p(n) z^{s_1 n} \right)^{\ell_1} \cdots \left( \sum_{n=0}^{\infty} p(n) z^{s_k n} \right)^{\ell_k} \\
&= \sum_{u_{1,1}=0}^{\infty} p(u_{1,1}) z^{s_1 u_{1,1}} \cdots \sum_{u_{1,\ell_1}=0}^{\infty} p(u_{1,\ell_1}) z^{s_1 u_{1,\ell_1}} \cdots \\
&\quad \sum_{u_{k,1}=0}^{\infty} p(u_{k,1}) z^{s_k u_{k,1}} \cdots \sum_{u_{k,\ell_k}=0}^{\infty} p(u_{k,\ell_k}) z^{s_k u_{k,\ell_k}} \\
&= \sum_{n=0}^{\infty} \left( \sum_{\mathbf{u} \in U_n} \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i}} p(u_{i,j}) \right) z^n.
\end{aligned}$$

Now Lemma 2.1 follows immediately.  $\square$

We will divide  $U_n$  into two parts  $U'_n$  and  $U''_n$  which will be given later such that  $u_{i,j} \rightarrow +\infty$  as  $n \rightarrow +\infty$  for any  $\mathbf{u} = (u_{1,1}, \dots, u_{1,\ell_1}, \dots, u_{k,1}, \dots, u_{k,\ell_k}) \in U'_n$ . By Lemma 2.1, we have

$$g(n) = \sum_{\mathbf{u} \in U'_n} \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i}} p(u_{i,j}) + \sum_{\mathbf{u} \in U''_n} \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i}} p(u_{i,j}). \quad (2.3)$$

We expect that  $U'_n$  contributes to  $g(n)$  the main term and  $U''_n$  contributes to  $g(n)$  the remainder term.

By (2.2), we have

$$\begin{aligned}
&\sum_{\mathbf{u} \in U'_n} \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i}} p(u_{i,j}) \\
&= \sum_{\mathbf{u} \in U'_n} \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i}} \frac{c_2 + O(u_{i,j}^{-1/2})}{u_{i,j}} \exp(c_1 \sqrt{u_{i,j}}) \\
&= \sum_{\mathbf{u} \in U'_n} \frac{c_2^{\ell_1 + \dots + \ell_k} + O(\sum u_{i,j}^{-1/2})}{\prod u_{i,j}} \exp\left(c_1 \sum_{i,j} \sqrt{u_{i,j}}\right) \quad (2.4)
\end{aligned}$$

and

$$\sum_{\mathbf{u} \in U''_n} \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i}} p(u_{i,j}) \ll \sum_{\mathbf{u} \in U''_n} \exp\left(c_1 \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i}} \sqrt{u_{i,j}}\right). \quad (2.5)$$

Since the function  $\exp(x)$  increases rapidly, it infers from (2.4) and (2.5) that the maximal value of  $\sum_{i,j} \sqrt{u_{i,j}}$  with  $\mathbf{u} \in U_n$  gives the main contribution to  $g(n)$ . Now we find the maximal value of  $\sum_{i,j} \sqrt{u_{i,j}}$  with  $\mathbf{u} \in U_n$ . By the Cauchy-Schwarz inequality, for any  $\mathbf{u} \in U_n$ , we have

$$\begin{aligned} \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i}} \sqrt{u_{i,j}} &= \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i}} \frac{1}{\sqrt{s_i}} \sqrt{s_i u_{i,j}} \\ &\leq \left( \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i}} \frac{1}{s_i} \right)^{1/2} \left( \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i}} s_i u_{i,j} \right)^{1/2} \\ &= \left( \sum_{i=1}^k \frac{\ell_i}{s_i} \right)^{1/2} \sqrt{n}, \end{aligned}$$

where the equality holds if and only if

$$u_{i,j} = \frac{n}{s_i^2 a}, \quad 1 \leq i \leq k, 1 \leq j \leq \ell_i,$$

where

$$a = \sum_{i=1}^k \frac{\ell_i}{s_i}.$$

Let

$$v_{i,j} = \frac{n}{s_i^2 a}, \quad 1 \leq i \leq k, 1 \leq j \leq \ell_i.$$

It is clear that

$$\sum_{i,j} s_i v_{i,j} = \sum_{i=1}^k \frac{\ell_i n}{s_i a} = n.$$

Now we have proved that the maximal value of  $\sum_{i,j} \sqrt{u_{i,j}}$  is obtained if and only if

$$\mathbf{u} = (u_{1,1}, \dots, u_{1,\ell_1}, \dots, u_{k,1}, \dots, u_{k,\ell_k}) = (v_{1,1}, \dots, v_{1,\ell_1}, \dots, v_{k,1}, \dots, v_{k,\ell_k}) := \mathbf{v}.$$

Basing on the above intuition that the maximal value of  $\sum_{i,j} \sqrt{u_{i,j}}$  with  $\mathbf{u} \in U_n$  gives the main contribution to  $g(n)$ , we take  $U'_n$  to be the set of  $\mathbf{u} \in U_n$  for which every  $u_{i,j}$  is near to  $v_{i,j}$  and  $U''_n$  to be the set of  $\mathbf{u} \in U_n$  for which  $u_{i,j}$  is far from  $v_{i,j}$  for some pair  $i, j$ . Since

$$s_1 u_{1,1} + \dots + s_1 u_{1,\ell_1} + \dots + s_k u_{k,1} + \dots + s_k u_{k,\ell_k} = n,$$

it is enough to take  $U'_n$  to be those  $\mathbf{u} \in U_n$  for which  $u_{i,j}$  is near to  $v_{i,j}$  for all  $i, j$  with  $i + j > 2$  and  $U''_n$  to be those  $\mathbf{u} \in U_n$  for which  $u_{i,j}$  is far from  $v_{i,j}$  for some pair  $i, j$  with  $i + j > 2$ . Now we give explicit  $U'_n$  and  $U''_n$ .

We appoint a real number  $\eta$  such that

$$\frac{3}{4} < \eta < \min \left\{ \frac{5}{6}, \frac{3}{4} \frac{\ell_1 + \cdots + \ell_k - 1}{\ell_1 + \cdots + \ell_k - 2} \right\}.$$

Since the right hand side is more than  $3/4$ , it follows that such  $\eta$  exists. Since  $3/4 < \eta < 5/6$ , we have  $2\eta - 3/2 > 0$  and  $3\eta - 5/2 < 0$ .

Let

$$U'_n = \{\mathbf{u} \in U_n : |u_{i,j} - v_{i,j}| < v_{i,j}^\eta \text{ for all pairs } i, j \text{ with } i + j > 2\}$$

and

$$U''_n = \{\mathbf{u} \in U_n : |u_{i,j} - v_{i,j}| \geq v_{i,j}^\eta \text{ for some pair } i, j \text{ with } i + j > 2\}.$$

We hope that  $U'_n$  contributes to  $g(n)$  the main term and  $U''_n$  contributes to  $g(n)$  the remainder term. These will be proved in the next section.

### 3 Preliminary Lemmas

Firstly we prove that  $U''_n$  contributes to  $g(n)$  the remainder term.

**Lemma 3.1.** *We have*

$$\sum_{\mathbf{u} \in U''_n} \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i}} p(u_{i,j}) = O \left( \exp \left( c_1 \sqrt{an} - c_3 n^{2\eta-3/2} \right) \right),$$

where  $c_1$  is given by (2.1) and  $c_3$  is a positive constant.

*Proof.* Let  $\mathbf{u} \in U''_n$ . Without loss of generality, we assume that

$$|u_{k,\ell_k} - v_{k,\ell_k}| \geq v_{k,\ell_k}^\eta. \quad (3.1)$$

Let

$$u_{k,\ell_k} = v_{k,\ell_k} + \alpha v_{k,\ell_k}.$$

Then  $|\alpha| \geq v_{k,\ell_k}^{-(1-\eta)}$ . Since  $\mathbf{u} \in U_n''$ , it follows that  $n \geq s_k u_{k,\ell_k}$ . Noting that  $n = s_k^2 a v_{k,\ell_k}$ , we have

$$v_{k,\ell_k} + \alpha v_{k,\ell_k} = u_{k,\ell_k} \leq \frac{n}{s_k} = s_k a v_{k,\ell_k}.$$

It follows that  $1 + \alpha \leq s_k a$ . So  $(s_k a - 1)^{-1} \alpha \leq 1$ .

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i}} \sqrt{u_{i,j}} &= \sqrt{u_{k,\ell_k}} + \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i, (i,j) \neq (k,\ell_k)}} \frac{1}{\sqrt{s_i}} \sqrt{s_i u_{i,j}} \\ &\leq \sqrt{u_{k,\ell_k}} + \left( \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i, (i,j) \neq (k,\ell_k)}} \frac{1}{s_i} \right)^{1/2} \left( \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i, (i,j) \neq (k,\ell_k)}} s_i u_{i,j} \right)^{1/2} \\ &= \sqrt{u_{k,\ell_k}} + \left( a - \frac{1}{s_k} \right)^{1/2} \sqrt{n - s_k u_{k,\ell_k}} \\ &= (v_{k,\ell_k} + \alpha v_{k,\ell_k})^{1/2} \\ &\quad + \left( a - \frac{1}{s_k} \right)^{1/2} (s_k^2 a v_{k,\ell_k} - s_k v_{k,\ell_k} - \alpha s_k v_{k,\ell_k})^{1/2} \\ &= f(\alpha) \sqrt{v_{k,\ell_k}}, \end{aligned}$$

where

$$f(x) = (1+x)^{1/2} + (s_k a - 1) (1 - (s_k a - 1)^{-1} x)^{1/2}.$$

Since

$$f'(x) = \frac{1}{2}(1+x)^{-1/2} - \frac{1}{2} (1 - (s_k a - 1)^{-1} x)^{-1/2},$$

it follows that  $f'(x) > 0$  for  $x < 0$  and  $f'(x) < 0$  for  $x > 0$ . By  $|\alpha| \geq v_{k,\ell_k}^{-(1-\eta)}$ , we have

$$f(\alpha) \leq \max\{f(-v_{k,\ell_k}^{-(1-\eta)}), f(v_{k,\ell_k}^{-(1-\eta)})\}.$$

For  $\beta \in \{-v_{k,\ell_k}^{-(1-\eta)}, v_{k,\ell_k}^{-(1-\eta)}\}$ , we have

$$\begin{aligned} f(\beta) &= 1 + \frac{1}{2}\beta - \frac{1}{8}\beta^2 + O(\beta^3) \\ &\quad + (s_k a - 1) \left( 1 - \frac{1}{2}(s_k a - 1)^{-1}\beta - \frac{1}{8}(s_k a - 1)^{-2}\beta^2 + O(\beta^3) \right) \\ &= s_k a - \frac{s_k a}{8(s_k a - 1)}\beta^2 + O(\beta^3) \\ &= s_k a - \frac{s_k a}{8(s_k a - 1)}v_{k,\ell_k}^{-2(1-\eta)} + O(v_{k,\ell_k}^{-3(1-\eta)}). \end{aligned}$$



Hence

$$\begin{aligned} \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i}} \sqrt{u_{i,j}} &\leq s_k a \sqrt{v_{k,\ell_k}} - \frac{s_k a}{8(s_k a - 1)} v_{k,\ell_k}^{-2(1-\eta)+1/2} + O(v_{k,\ell_k}^{-3(1-\eta)+1/2}) \\ &\leq \sqrt{an} - \delta n^{2\eta-3/2} \end{aligned}$$

for all sufficiently large integers  $n$ , where  $\delta$  is a positive constant. Thus, noting that  $s_k^2 a v_{k,\ell_k} = n$ , for any  $\mathbf{u} \in U_n''$ , and by (2.5), we have

$$\begin{aligned} &\sum_{\mathbf{u} \in U_n''} \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i}} p(u_{i,j}) \\ &\ll \sum_{\mathbf{u} \in U_n''} \exp \left( c_1 \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i}} \sqrt{u_{i,j}} \right) \\ &\ll \sum_{\mathbf{u} \in U_n''} \exp \left( c_1 \sqrt{an} - c_1 \delta n^{2\eta-3/2} \right) \\ &\ll (n+1)^{\ell_1+\dots+\ell_k-1} \exp \left( c_1 \sqrt{an} - c_1 \delta n^{2\eta-3/2} \right) \\ &\ll \exp \left( c_1 \sqrt{an} - c_3 n^{2\eta-3/2} \right), \end{aligned}$$

for all sufficiently large integers  $n$ , where  $c_3$  is a positive constant. This completes the proof of Lemma 3.1.  $\square$

Lemma 3.1 deals with those  $\mathbf{u}$  for which  $u_{i,j}$  is far from  $v_{i,j}$  for some pair  $i, j$  with  $i+j > 2$ . These  $\mathbf{u}$  contribute to  $g(n)$  with the remainder. Now we deal with all  $\mathbf{u}$  for which  $u_{i,j}$  is near to  $v_{i,j}$  for every pair  $i, j$  with  $i+j > 2$ .

**Lemma 3.2.** *We have*

$$\sum_{\mathbf{u} \in U_n'} \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i}} p(u_{i,j}) = \frac{c_2^{\ell_1+\dots+\ell_k} + O(n^{\eta-1})}{\prod v_{i,j}} \sum_{\mathbf{u} \in U_n'} \exp \left( c_1 \sum_{i,j} \sqrt{u_{i,j}} \right),$$

where  $c_1$  and  $c_2$  are given by (2.1).

*Proof.* Recall that

$$U_n' = \{\mathbf{u} \in U_n : |u_{i,j} - v_{i,j}| < v_{i,j}^\eta \text{ for all } i+j > 2\}.$$

Let  $\mathbf{u} \in U_n''$ . By  $\sum_{i,j} s_i v_{i,j} = n$  and  $|u_{i,j} - v_{i,j}| < v_{i,j}^\eta$ , we have

$$|u_{1,1} - v_{1,1}| = \left| \sum_{i+j>2} s_i (u_{i,j} - v_{i,j}) \right| < \sum_{i+j>2} s_i v_{i,j}^\eta < \sum_{i+j>2} s_i n^\eta.$$

Thus  $u_{i,j} \rightarrow +\infty$  as  $n \rightarrow +\infty$  for  $i+j \geq 2$ . Moreover, by  $v_{i,j} = O(n)$ , we have  $|u_{i,j} - v_{i,j}| \ll n^\eta$  for all  $i, j$  with  $i+j \geq 2$ . Thus  $u_{i,j} = O(n)$  all  $i, j$  with  $i+j \geq 2$ . Hence

$$\frac{1}{u_{i,j}} = \frac{1}{v_{i,j} + O(n^\eta)} = \frac{1}{v_{i,j}(1 + O(n^{\eta-1}))} = \frac{1}{v_{i,j}}(1 + O(n^{\eta-1})), \quad i+j \geq 2$$

and

$$\sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i}} u_{i,j}^{-1/2} = O(n^{-1/2}).$$

Noting that

$$\prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i}} (1 + O(n^{\eta-1})) = 1 + O(n^{\eta-1}),$$

from the above arguments, (2.4) and  $\frac{3}{4} < \eta < \frac{5}{6}$ , we have

$$\begin{aligned} & \sum_{\mathbf{u} \in U'_n} \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i}} p(u_{i,j}) \\ &= \frac{c_2^{\ell_1 + \dots + \ell_k} + O(n^{-1/2})}{\prod v_{i,j}} (1 + O(n^{\eta-1})) \sum_{\mathbf{u} \in U'_n} \exp\left(c_1 \sum_{i,j} \sqrt{u_{i,j}}\right) \\ &= \frac{c_2^{\ell_1 + \dots + \ell_k} + O(n^{\eta-1})}{\prod v_{i,j}} \sum_{\mathbf{u} \in U'_n} \exp\left(c_1 \sum_{i,j} \sqrt{u_{i,j}}\right). \end{aligned}$$

□

The following two lemmas devote to convert summations on integral variables into integrals.

**Lemma 3.3.** *Suppose that  $f$  is a function on  $[a, b]$  such that  $f'$  exists with  $m$  zero points on  $(a, b)$ . Then*

$$\sum_{a \leq n \leq b} f(n) = \int_a^b f(x) dx + O\left((m+1) \max_{a \leq x \leq b} |f(x)|\right).$$

*Proof.* We divide  $[a, b]$  into  $m+1$  intervals  $[a_0, a_1], [a_1, a_2], \dots, [a_m, a_{m+1}]$  such that  $f' \neq 0$  on  $(a_i, a_{i+1})$ . Given  $0 \leq i \leq m$ . Without loss of generality, we assume that  $f' > 0$  on  $(a_i, a_{i+1})$ . For any integer  $n \in (a_i + 1, a_{i+1} - 1)$ , we have

$$\int_{n-1}^n f(x) dx \leq f(n) \leq \int_n^{n+1} f(x) dx.$$

Summing on  $n$ , we have

$$\left| \sum_{a_i \leq n \leq a_{i+1}} f(n) - \int_{a_i}^{a_{i+1}} f(x) dx \right| = O \left( \max_{a_i \leq x \leq a_{i+1}} |f(x)| \right) = O \left( \max_{a \leq x \leq b} |f(x)| \right).$$

Then

$$\begin{aligned} \left| \sum_{a \leq n \leq b} f(n) - \int_a^b f(x) dx \right| &\leq \sum_{i=0}^m \left| \sum_{a_i \leq n \leq a_{i+1}} f(n) - \int_{a_i}^{a_{i+1}} f(x) dx \right| \\ &= O \left( (m+1) \max_{a \leq x \leq b} |f(x)| \right). \end{aligned}$$

This completes the proof.  $\square$

**Lemma 3.4.** *Suppose that  $f$  is a function on*

$$D = \{(x_1, \dots, x_s) : a_i \leq x_i \leq a_i + b_i, 1 \leq i \leq s\}$$

*such that, for each  $i$  and fixed  $\{x_1, \dots, x_s\} \setminus \{x_i\}$ , the partial derivative  $\frac{\partial}{\partial x_i} f$  exists with at most  $m_i$  zero points on  $[a_i, a_i + b_i]$ . Then*

$$\begin{aligned} &\sum_{(n_1, \dots, n_s) \in D \cap \mathbb{Z}^s} f(n_1, \dots, n_s) \\ &= \int_D \cdots \int f(x_1, \dots, x_s) dx_1 \cdots dx_s + O \left( \sum_{1 \leq i \leq s} \frac{m_i + 1}{b_i} BM \right), \end{aligned}$$

*where  $B = b_1 b_2 \cdots b_s$  and  $M = \max_D |f(x_1, \dots, x_s)|$ .*

*Proof.* Let  $c_i = a_i + b_i(1 \leq i \leq s)$ . By Lemma 3.3, we have

$$\begin{aligned}
& \sum_{(n_1, \dots, n_s) \in D \cap \mathbb{Z}^s} f(n_1, \dots, n_s) \\
&= \sum_{\substack{a_2 \leq n_2 \leq c_2 \\ \dots \\ a_s \leq n_s \leq c_s}} \left( \int_{a_1}^{c_1} f(x_1, n_2, \dots, n_s) dx_1 + O((m_1 + 1)M) \right) \\
&= \sum_{\substack{a_2 \leq n_2 \leq c_2 \\ \dots \\ a_s \leq n_s \leq c_s}} \int_{a_1}^{c_1} f(x_1, n_2, \dots, n_s) dx_1 + O\left(\frac{m_1 + 1}{b_1} BM\right) \\
&= \sum_{\substack{a_3 \leq n_3 \leq c_3 \\ \dots \\ a_s \leq n_s \leq c_s}} \left( \int_{a_2}^{c_2} \int_{a_1}^{c_1} f(x_1, x_2, n_3, \dots, n_s) dx_1 dx_2 + O((m_2 + 1)b_1 M) \right) \\
&\quad + O\left(\frac{m_1 + 1}{b_1} BM\right) \\
&= \sum_{\substack{a_3 \leq n_3 \leq c_3 \\ \dots \\ a_s \leq n_s \leq c_s}} \int_{a_2}^{c_2} \int_{a_1}^{c_1} f(x_1, x_2, n_3, \dots, n_s) dx_1 dx_2 + O\left(\left(\frac{m_1 + 1}{b_1} + \frac{m_2 + 1}{b_2}\right) BM\right) \\
&\quad \vdots \\
&= \int_D \dots \int f(x_1, \dots, x_s) dx_1 \dots dx_s + O\left(\sum_{1 \leq i \leq s} \frac{m_i + 1}{b_i} BM\right).
\end{aligned}$$

This completes the proof of Lemma 3.4.  $\square$

**Lemma 3.5.** *We have*

$$\begin{aligned}
\sum_{\mathbf{u} \in U'_n} \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i}} p(u_{i,j}) &= \frac{c_2^{\ell_1 + \dots + \ell_k} + O(n^{\eta-1})}{\prod v_{i,j}} \int_{D_1} \dots \int \exp\left(c_1 \sum_{i,j} \sqrt{x_{i,j}}\right) dx_{1,2} \dots dx_{k,\ell_k} \\
&\quad + O\left(n^{-(1-\eta)(\ell_1 + \dots + \ell_k) - 2\eta} \exp\left(c_1 \sqrt{an}\right)\right),
\end{aligned}$$

where  $c_1$  and  $c_2$  are given by (2.1), and  $D_1$  is the set of all  $\ell_1 + \dots + \ell_k - 1$  tuples  $(x_{1,2}, \dots, x_{k,\ell_k})$  of real numbers with

$$|x_{i,j} - v_{i,j}| \leq v_{i,j}^\eta \quad (3.2)$$

for all  $i + j > 2$ , and

$$x_{1,1} = n - \sum_{i+j>2} s_i x_{i,j}.$$

*Proof.* By Lemma 3.2, we have

$$\sum_{\mathbf{u} \in U'_n} \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i}} p(u_{i,j}) = \frac{c_2^{\ell_1 + \dots + \ell_k} + O(n^{\eta-1})}{\prod v_{i,j}} \sum_{\mathbf{u} \in U'_n} \exp \left( c_1 \sum_{i,j} \sqrt{u_{i,j}} \right).$$

For the function

$$f(x_{1,2}, \dots, x_{1,\ell_1}, \dots, x_{k,1}, \dots, x_{k,\ell_k}) = \exp \left( \sqrt{n - \sum_{i+j>2} s_i x_{i,j}} + \sum_{i+j>2} \sqrt{x_{i,j}} \right),$$

by Lemma 3.4, we have

$$\begin{aligned} & \sum_{\mathbf{u} \in U'_n} \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_i}} p(u_{i,j}) \\ &= \frac{c_2^{\ell_1 + \dots + \ell_k} + O(n^{\eta-1})}{\prod v_{i,j}} \int \cdots \int_{D_1} \exp \left( c_1 \sum_{i,j} \sqrt{x_{i,j}} \right) dx_{1,2} \cdots dx_{k,\ell_k} \\ & \quad + O \left( n^{-(1-\eta)(\ell_1 + \dots + \ell_k) - 2\eta} \max_{D_1} \exp \left( c_1 \sum_{i,j} \sqrt{x_{i,j}} \right) \right). \end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \sum_{i,j} \sqrt{x_{i,j}} &= \sum_{i,j} \frac{1}{\sqrt{s_i}} \sqrt{s_i x_{i,j}} \\ &\leq \left( \sum_{i,j} \frac{1}{s_i} \right)^{1/2} \left( \sum_{i,j} s_i x_{i,j} \right)^{1/2} \\ &= \sqrt{an}. \end{aligned}$$

Thus, we have

$$\begin{aligned} & O \left( n^{-(1-\eta)(\ell_1 + \dots + \ell_k) - 2\eta} \max_{D_1} \exp \left( c_1 \sum_{i,j} \sqrt{x_{i,j}} \right) \right) \\ &= O \left( n^{-(1-\eta)(\ell_1 + \dots + \ell_k) - 2\eta} \exp \left( c_1 \sqrt{an} \right) \right). \end{aligned}$$

□

**Lemma 3.6.** *We have*

$$\begin{aligned} g(n) &= \frac{c_2^{\ell_1 + \dots + \ell_k} + O(n^{3\eta-5/2})}{v_{1,1}} \left( \frac{8}{c_1} \right)^{(\ell_1 + \dots + \ell_k - 1)/2} \left( \frac{v_{1,1}}{\prod v_{i,j}} \right)^{1/4} \exp(c_1 \sqrt{an}) \\ & \quad \sqrt{s_1}^{\ell_1-1} \sqrt{s_2}^{\ell_2} \cdots \sqrt{s_k}^{\ell_k} \cdot \int \cdots \int_{\Omega} \exp \left( - \sum_{i,j} s_i w_{i,j}^2 \right) dw_{1,2} \cdots dw_{k,\ell_k} \\ & \quad + O \left( n^{-(1-\eta)(\ell_1 + \dots + \ell_k) - 2\eta} \exp \left( c_1 \sqrt{an} \right) \right), \end{aligned}$$

where  $\Omega$  is the set of all  $\ell_1 + \dots + \ell_k - 1$  tuples  $(w_{1,2}, \dots, w_{k,\ell_k})$  of real numbers with  $|w_{i,j}| < \sqrt{\frac{c_1}{8s_i}} v_{i,j}^{\eta-3/4}$  for all  $i+j > 2$ , and  $w_{1,1} = -\sum_{i+j>2} w_{i,j}$ .

*Proof.* Noting that

$$\exp(c_1\sqrt{an} - c_2n^{2\eta-3/2}) = O\left(n^{-(1-\eta)(\ell_1+\dots+\ell_k)-2\eta} \exp(c_1\sqrt{an})\right),$$

by (2.3), Lemmas 3.1 and 3.5, we have

$$\begin{aligned} g(n) &= \frac{c_2^{\ell_1+\dots+\ell_k} + O(n^{\eta-1})}{\prod v_{i,j}} \int_{D_1} \dots \int \exp\left(c_1 \sum_{i,j} \sqrt{x_{i,j}}\right) dx_{1,2} \dots dx_{k,\ell_k} \\ &\quad + O\left(n^{-(1-\eta)(\ell_1+\dots+\ell_k)-2\eta} \exp(c_1\sqrt{an})\right). \end{aligned} \quad (3.3)$$

Recall that  $D_1$  is the set of all  $\ell_1 + \dots + \ell_k - 1$  tuples  $(x_{1,2}, \dots, x_{k,\ell_k})$  of real numbers with

$$|x_{i,j} - v_{i,j}| \leq v_{i,j}^\eta$$

for all  $i+j > 2$ , and

$$x_{1,1} = n - \sum_{i+j>2} s_i x_{i,j}.$$

Let  $x_{i,j} = v_{i,j} + v_{i,j} y_{i,j}$  for all  $i, j$ . Then

$$\begin{aligned} &\int_{D_1} \dots \int \exp\left(c_1 \sum_{i,j} \sqrt{x_{i,j}}\right) dx_{1,2} \dots dx_{k,\ell_k} \\ &= \frac{\prod v_{i,j}}{v_{1,1}} \int_{D_2} \dots \int \exp\left(c_1 \sum_{i,j} \sqrt{v_{i,j} + v_{i,j} y_{i,j}}\right) dy_{1,2} \dots dy_{k,\ell_k}, \end{aligned}$$

where  $D_2$  is the set of all  $\ell_1 + \dots + \ell_k - 1$  tuples  $(y_{1,2}, \dots, y_{k,\ell_k})$  of real numbers with

$$|y_{i,j}| \leq v_{i,j}^{-(1-\eta)} \quad (3.4)$$

for all  $i+j > 2$ , and

$$s_1 v_{1,1} y_{1,1} = - \sum_{i+j>2} s_i v_{i,j} y_{i,j}. \quad (3.5)$$

The last equality comes from  $s_1 = 1$  and

$$\sum_{i,j} s_i x_{i,j} = n = \sum_{i,j} s_i v_{i,j}.$$

By (3.4) and (3.5), we have

$$|y_{i,j}| \ll n^{-(1-\eta)} \quad (3.6)$$

for all  $i, j$ . By the definition of  $v_{i,j}$ , we have

$$s_i v_{i,j} = \frac{n}{s_i a}.$$

Thus, by (3.5), we have

$$\sum_{i,j} \frac{n}{s_i a} y_{i,j} = 0.$$

That is,

$$\sum_{i,j} \frac{y_{i,j}}{s_i} = 0.$$

Hence

$$\sum_{i,j} \sqrt{v_{i,j}} y_{i,j} = \sum_{i,j} \sqrt{\frac{n}{s_i^2 a}} y_{i,j} = \sqrt{\frac{n}{a}} \sum_{i,j} \frac{y_{i,j}}{s_i} = 0.$$

Thus

$$\begin{aligned} & \sum_{i,j} \sqrt{v_{i,j} + v_{i,j} y_{i,j}} \\ &= \sum_{i,j} \sqrt{v_{i,j}} \left( 1 + \frac{1}{2} y_{i,j} - \frac{1}{8} y_{i,j}^2 + O(y_{i,j}^3) \right) \\ &= \sum_{i,j} \sqrt{v_{i,j}} - \frac{1}{8} \sum_{i,j} \sqrt{v_{i,j}} y_{i,j}^2 + O(n^{3\eta-5/2}) \\ &= \sum_{i,j} \sqrt{\frac{n}{s_i^2 a}} - \frac{1}{8} \sum_{i,j} \sqrt{v_{i,j}} y_{i,j}^2 + O(n^{3\eta-5/2}) \\ &= \sqrt{an} - \frac{1}{8} \sum_{i,j} \sqrt{v_{i,j}} y_{i,j}^2 + O(n^{3\eta-5/2}). \end{aligned}$$

Therefore

$$\begin{aligned} & \int \cdots \int_{D_2} \exp \left( c_1 \sum_{i,j} \sqrt{v_{i,j} + v_{i,j} y_{i,j}} \right) dy_{1,2} \cdots dy_{k,\ell_k} \\ &= \exp(O(n^{3\eta-5/2})) \exp(c_1 \sqrt{an}) \int \cdots \int_{D_2} \exp \left( -\frac{c_1}{8} \sum_{i,j} \sqrt{v_{i,j}} y_{i,j}^2 \right) dy_{1,2} \cdots dy_{k,\ell_k} \\ &= (1 + O(n^{3\eta-5/2})) \exp(c_1 \sqrt{an}) \int \cdots \int_{D_2} \exp \left( -\frac{c_1}{8} \sum_{i,j} \sqrt{v_{i,j}} y_{i,j}^2 \right) dy_{1,2} \cdots dy_{k,\ell_k}. \end{aligned}$$

Let

$$v_{i,j}^{1/4} y_{i,j} = z_{i,j}, \quad \sqrt{\frac{c_1}{8}} z_{i,j} = \sqrt{s_i} w_{i,j}$$

for all  $i, j$ . Then

$$\begin{aligned} & \int \cdots \int_{D_2} \exp \left( -\frac{c_1}{8} \sum_{i,j} \sqrt{v_{i,j}} y_{i,j}^2 \right) dy_{1,2} \cdots dy_{k,\ell_k} \\ &= \left( \frac{v_{1,1}}{\prod v_{i,j}} \right)^{1/4} \int \cdots \int_{D_3} \exp \left( -\frac{c_1}{8} \sum_{i,j} z_{i,j}^2 \right) dz_{1,2} \cdots dz_{k,\ell_k} \\ &= \left( \frac{v_{1,1}}{\prod v_{i,j}} \right)^{1/4} \left( \frac{8}{c_1} \right)^{(\ell_1 + \cdots + \ell_k - 1)/2} \sqrt{s_1}^{\ell_1 - 1} \sqrt{s_2}^{\ell_2} \cdots \sqrt{s_k}^{\ell_k} \\ & \quad \cdot \int \cdots \int_{\Omega} \exp \left( -\sum_{i,j} s_i w_{i,j}^2 \right) dw_{1,2} \cdots dw_{k,\ell_k}, \end{aligned}$$

where  $D_3$  is the set of all  $\ell_1 + \cdots + \ell_k - 1$  tuples  $(z_{1,2}, \dots, z_{k,\ell_k})$  of real numbers with

$$|z_{i,j}| \leq v_{i,j}^{\eta-3/4}$$

for all  $i + j > 2$ , and

$$s_1 v_{1,1}^{3/4} z_{1,1} = - \sum_{i+j>2} s_i v_{i,j}^{3/4} z_{i,j},$$

$\Omega$  is the set of all  $\ell_1 + \cdots + \ell_k - 1$  tuples  $(w_{1,2}, \dots, w_{k,\ell_k})$  of real numbers with

$$|w_{i,j}| < \sqrt{\frac{c_1}{8s_i}} v_{i,j}^{\eta-3/4}$$

for all  $i + j > 2$ , and

$$s_1^{3/2} v_{1,1}^{3/4} w_{1,1} = - \sum_{i+j>2} s_i^{3/2} v_{i,j}^{3/4} w_{i,j}. \quad (3.7)$$

Noting that  $s_i^{3/2} v_{i,j}^{3/4} = (n/a)^{3/4}$ , (3.7) is equivalent to

$$w_{1,1} = - \sum_{i+j>2} w_{i,j}. \quad (3.8)$$



Hence

$$\begin{aligned}
& \int \cdots \int_{D_1} \exp \left( c_1 \sum_{i,j} \sqrt{x_{i,j}} \right) dx_{1,2} \cdots dx_{k,\ell_k} \\
&= \frac{\prod v_{i,j}}{v_{1,1}} \int \cdots \int_{D_2} \exp \left( c_1 \sum_{i,j} \sqrt{v_{i,j} + v_{i,j} y_{i,j}} \right) dy_{1,2} \cdots dy_{k,\ell_k} \\
&= \frac{\prod v_{i,j}}{v_{1,1}} (1 + O(n^{3\eta-5/2})) \exp(c_1 \sqrt{an}) \int \cdots \int_{D_2} \exp \left( -\frac{c_1}{8} \sum_{i,j} \sqrt{v_{i,j}} y_{i,j}^2 \right) dy_{1,2} \cdots dy_{k,\ell_k} \\
&= \frac{\prod v_{i,j}}{v_{1,1}} (1 + O(n^{3\eta-5/2})) \exp(c_1 \sqrt{an}) \left( \frac{v_{1,1}}{\prod v_{i,j}} \right)^{1/4} \left( \frac{8}{c_1} \right)^{(\ell_1 + \cdots + \ell_k - 1)/2} \\
&\quad \sqrt{s_1}^{\ell_1-1} \sqrt{s_2}^{\ell_2} \cdots \sqrt{s_k}^{\ell_k} \cdot \int \cdots \int_{\Omega} \exp \left( -\sum_{i,j} s_i w_{i,j}^2 \right) dw_{1,2} \cdots dw_{k,\ell_k}.
\end{aligned}$$

It follows from (3.3) and  $\frac{3}{4} < \eta < \frac{5}{6}$  that

$$\begin{aligned}
g(n) &= \frac{c_2^{\ell_1 + \cdots + \ell_k} + O(n^{\eta-1})}{\prod v_{i,j}} \int \cdots \int_{D_1} \exp \left( c_1 \sum_{i,j} \sqrt{x_{i,j}} \right) dx_{1,2} \cdots dx_{k,\ell_k} \\
&\quad + O(n^{-(1-\eta)(\ell_1 + \cdots + \ell_k) - 2\eta} \exp(c_1 \sqrt{an})) \\
&= \frac{c_2^{\ell_1 + \cdots + \ell_k} + O(n^{\eta-1})}{\prod v_{i,j}} \frac{\prod v_{i,j}}{v_{1,1}} (1 + O(n^{3\eta-5/2})) \exp(c_1 \sqrt{an}) \left( \frac{v_{1,1}}{\prod v_{i,j}} \right)^{1/4} \\
&\quad \left( \frac{8}{c_1} \right)^{(\ell_1 + \cdots + \ell_k - 1)/2} \sqrt{s_1}^{\ell_1-1} \sqrt{s_2}^{\ell_2} \cdots \sqrt{s_k}^{\ell_k} \\
&\quad \cdot \int \cdots \int_{\Omega} \exp \left( -\sum_{i,j} s_i w_{i,j}^2 \right) dw_{1,2} \cdots dw_{k,\ell_k} \\
&\quad + O(n^{-(1-\eta)(\ell_1 + \cdots + \ell_k) - 2\eta} \exp(c_1 \sqrt{an})) \\
&= \frac{c_2^{\ell_1 + \cdots + \ell_k} + O(n^{3\eta-5/2})}{v_{1,1}} \left( \frac{8}{c_1} \right)^{(\ell_1 + \cdots + \ell_k - 1)/2} \left( \frac{v_{1,1}}{\prod v_{i,j}} \right)^{1/4} \exp(c_1 \sqrt{an}) \\
&\quad \sqrt{s_1}^{\ell_1-1} \sqrt{s_2}^{\ell_2} \cdots \sqrt{s_k}^{\ell_k} \cdot \int \cdots \int_{\Omega} \exp \left( -\sum_{i,j} s_i w_{i,j}^2 \right) dw_{1,2} \cdots dw_{k,\ell_k} \\
&\quad + O(n^{-(1-\eta)(\ell_1 + \cdots + \ell_k) - 2\eta} \exp(c_1 \sqrt{an})).
\end{aligned}$$

□

Now we determine the value of integral in Lemma 3.6

$$\int_{\Omega} \cdots \int \exp \left( - \sum_{i,j} s_i w_{i,j}^2 \right) dw_{1,2} \cdots dw_{k,\ell_k}.$$

To do this, we need the following general lemma. We believe it should appear in somewhere.

**Lemma 3.7.** *If  $\mathbf{x}^T A \mathbf{x}$  is a positive definite quadratic form in  $\mathbf{x}^T = (x_1, \dots, x_k)$  and  $U$  is a region in  $\mathbb{R}^k$ , then there is a linear transformation  $\mathbf{x} = C \mathbf{y}$  such that*

$$\int_U \cdots \int e^{-\mathbf{x}^T A \mathbf{x}} d\mathbf{x} = \frac{1}{\sqrt{\det(A)}} \int_{U'} \cdots \int e^{-y_1^2 - \cdots - y_k^2} d\mathbf{y},$$

where

$$U' = \{\mathbf{y} : \mathbf{y} = C^{-1} \mathbf{x}, \mathbf{x} \in U\}.$$

*Proof.* Let  $\mathbf{x} = C \mathbf{y}$  be a linear transformation such that  $C^T A C = I$  is the unit matrix. Then

$$|\det(C)| = \frac{1}{\sqrt{\det(A)}}.$$

Therefore

$$\begin{aligned} \int_{U'} \cdots \int e^{-\mathbf{x}^T A \mathbf{x}} d\mathbf{x} &= |\det(C)| \int_{U'} \cdots \int e^{-y_1^2 - \cdots - y_k^2} d\mathbf{y} \\ &= \frac{1}{\sqrt{\det(A)}} \int_{U'} \cdots \int e^{-y_1^2 - \cdots - y_k^2} d\mathbf{y}. \end{aligned}$$

□

**Lemma 3.8.** *Let  $a_0, a_1, \dots, a_k$  be  $k+1$  positive real numbers. Then there is a linear transformation  $\mathbf{x} = C \mathbf{y}$  such that*

$$\begin{aligned} &\int_U \cdots \int \exp \left( -a_0(x_1 + \cdots + x_k)^2 - a_1 x_1^2 - \cdots - a_k x_k^2 \right) d\mathbf{x} \\ &= \left( a_0 a_1 \cdots a_k \sum_{i=0}^k \frac{1}{a_i} \right)^{-1/2} \int_{U'} \cdots \int e^{-y_1^2 - \cdots - y_k^2} d\mathbf{y}, \end{aligned}$$

where

$$U' = \{\mathbf{y} : \mathbf{y} = C^{-1} \mathbf{x}, \mathbf{x} \in U\}.$$

*Proof.* It is clear that the quadratic form

$$a_0(x_1 + \cdots + x_k)^2 + a_1x_1^2 + \cdots + a_kx_k^2$$

is positive definite. Its matrix is

$$A_k = \begin{pmatrix} a_0 + a_1 & a_0 & a_0 & \cdots & a_0 \\ a_0 & a_0 + a_2 & a_0 & \cdots & a_0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_0 & a_0 & a_0 & \cdots & a_0 + a_k \end{pmatrix}.$$

It is not difficult to see that

$$\det(A_k) = a_k \det(A_{k-1}) + a_0 a_1 \cdots a_{k-1}.$$

By induction on  $k$ , we have

$$\det(A_k) = a_0 a_1 \cdots a_k \sum_{i=0}^k \frac{1}{a_i}.$$

Now Lemma 3.8 follows from Lemma 3.7. □

## 4 Proof of Theorem 1.1

It follows from Lemma 3.6 that

$$\begin{aligned} g(n) &= \frac{c_2^{\ell_1 + \cdots + \ell_k} + O(n^{3\eta - 5/2})}{v_{1,1}} \left( \frac{8}{c_1} \right)^{(\ell_1 + \cdots + \ell_k - 1)/2} \left( \frac{v_{1,1}}{\prod v_{i,j}} \right)^{1/4} \exp(c_1 \sqrt{an}) \\ &\quad \sqrt{s_1}^{\ell_1 - 1} \sqrt{s_2}^{\ell_2} \cdots \sqrt{s_k}^{\ell_k} \cdot \int \cdots \int_{\Omega} \exp \left( - \sum_{i,j} s_i w_{i,j}^2 \right) dw_{1,2} \cdots dw_{k,\ell_k} \\ &\quad + O \left( n^{-(1-\eta)(\ell_1 + \cdots + \ell_k) - 2\eta} \exp(c_1 \sqrt{an}) \right) \end{aligned}$$

Thus, we only need estimate  $\int \cdots \int_{\Omega} \exp \left( -\sum_{i,j} s_i w_{i,j}^2 \right) dw_{1,2} \cdots dw_{k,\ell_k}$ . By Lemma 3.8, we have

$$\begin{aligned} & \int \cdots \int_{\Omega} \exp \left( -\sum_{i,j} s_i w_{i,j}^2 \right) dw_{1,2} \cdots dw_{k,\ell_k} \\ &= \left( s_1^{\ell_1} s_2^{\ell_2} \cdots s_k^{\ell_k} \sum_{i=1}^k \frac{\ell_i}{s_i} \right)^{-1/2} \int \cdots \int_{\Omega'} \exp \left( -\sum_{i+j>2} w_{i,j}'^2 \right) dw_{1,2}' \cdots dw_{k,\ell_k}', \end{aligned}$$

where  $\Omega'$  is the set of all  $\ell_1 + \cdots + \ell_k - 1$  tuples  $(w_{1,2}', \dots, w_{k,\ell_k}')$  of real numbers with

$$|w_{i,j}'| < c_{i,j}' v_{i,j}^{\eta-3/4}$$

for all  $i+j > 2$  and some positive constants  $c_{i,j}'$ . Noting that, for  $x > 1$ ,

$$\int_x^\infty e^{-t^2} dt = \int_x^\infty e^{-t^2+t-t} dt \leq e^{-x^2+x} \int_x^\infty e^{-t} dt = e^{-x^2}$$

and the well known Gaussian integral (also known as the Euler Poisson integral)

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi},$$

for  $\eta > \frac{3}{4}$ , we have

$$\begin{aligned} \int_{-c_{i,j}' v_{i,j}^{\eta-3/4}}^{c_{i,j}' v_{i,j}^{\eta-3/4}} e^{-x^2} dx &= \int_{-c_{i,j}'' n^{\eta-3/4}}^{c_{i,j}'' n^{\eta-3/4}} e^{-x^2} dx \\ &= \int_{-\infty}^{\infty} e^{-x^2} dx + O(e^{-(c_{i,j}''^2 n^{2\eta-3/2})}) \\ &= \sqrt{\pi} + O(e^{-(c_{i,j}''^2 n^{2\eta-3/2})}), \end{aligned}$$

where  $c_{i,j}''$  are some positive constants. Thus

$$\begin{aligned} & \int \cdots \int_{\Omega'} \exp \left( -\sum_{i+j>2} w_{i,j}'^2 \right) dw_{1,2}' \cdots dw_{k,\ell_k}' \\ &= \left( \sqrt{\pi} + O \left( e^{-(c_{1,2}'^2 n^{2\eta-3/2})} \right) \right) \cdots \left( \sqrt{\pi} + O \left( e^{-(c_{k,\ell_k}'^2 n^{2\eta-3/2})} \right) \right) \\ &= \pi^{(\ell_1 + \cdots + \ell_k - 1)/2} + O \left( e^{-(c n^{2\eta-3/2})} \right), \end{aligned}$$

where  $c$  is a positive constant. Therefore,

$$\begin{aligned}
g(n) &= \frac{c_2^{\ell_1+\dots+\ell_k} + O(n^{3\eta-5/2})}{v_{1,1}} \left(\frac{8}{c_1}\right)^{(\ell_1+\dots+\ell_k-1)/2} \left(\frac{v_{1,1}}{\prod v_{i,j}}\right)^{1/4} \exp(c_1\sqrt{an}) \\
&\quad \cdot \sqrt{s_1}^{\ell_1-1} \sqrt{s_2}^{\ell_2} \dots \sqrt{s_k}^{\ell_k} \cdot \left(s_1^{\ell_1} s_2^{\ell_2} \dots s_k^{\ell_k} \sum_{i=1}^k \frac{\ell_i}{s_i}\right)^{-1/2} \\
&\quad \cdot (\pi^{(\ell_1+\dots+\ell_k-1)/2} + O(e^{-(cn^{2\eta-3/2})})) \\
&\quad + O\left(n^{-(1-\eta)(\ell_1+\dots+\ell_k)-2\eta} \exp(c_1\sqrt{an})\right) \\
&= \frac{c_2^{\ell_1+\dots+\ell_k} \pi^{(\ell_1+\dots+\ell_k-1)/2} + O(n^{3\eta-5/2})}{v_{1,1}} \left(\frac{8}{c_1}\right)^{(\ell_1+\dots+\ell_k-1)/2} \left(\frac{v_{1,1}}{\prod v_{i,j}}\right)^{1/4} \\
&\quad \exp(c_1\sqrt{an}) \sqrt{s_1}^{\ell_1-1} \sqrt{s_2}^{\ell_2} \dots \sqrt{s_k}^{\ell_k} \cdot \left(s_1^{\ell_1} s_2^{\ell_2} \dots s_k^{\ell_k} \sum_{i=1}^k \frac{\ell_i}{s_i}\right)^{-1/2} \\
&\quad + O\left(n^{-(1-\eta)(\ell_1+\dots+\ell_k)-2\eta} \exp(c_1\sqrt{an})\right) \\
&= 2^{-(3\ell_1+\dots+3\ell_k+5)/4} 3^{-(\ell_1+\dots+\ell_k+1)/4} a^{(\ell_1+\dots+\ell_k+1)/4} s_1^{\ell_1/2} \dots s_k^{\ell_k/2} \\
&\quad \cdot n^{-3/4-(\ell_1+\dots+\ell_k)/4} \exp(c_1\sqrt{an}) \\
&\quad + O\left(n^{3\eta-\frac{5}{2}-1-(\ell_1+\dots+\ell_k-1)/4} \exp(c_1\sqrt{an})\right) \\
&\quad + O\left(n^{-(1-\eta)(\ell_1+\dots+\ell_k)-2\eta} \exp(c_1\sqrt{an})\right) \\
&= c(\mathbf{s}, \mathbf{l}) n^{d(\mathbf{l})} \exp(c_1\sqrt{an}) \\
&\quad + O\left(n^{3\eta-\frac{7}{2}-(\ell_1+\dots+\ell_k-1)/4} \exp(c_1\sqrt{an})\right) \\
&\quad + O\left(n^{-(1-\eta)(\ell_1+\dots+\ell_k)-2\eta} \exp(c_1\sqrt{an})\right).
\end{aligned}$$

Noting that  $\eta$  is a real number that subjects to

$$\frac{3}{4} < \eta < \min \left\{ \frac{5}{6}, \frac{3}{4} \frac{\ell_1 + \dots + \ell_k - 1}{\ell_1 + \dots + \ell_k - 2} \right\},$$

we choose  $\eta = \frac{3}{4} + \varepsilon'$ , where  $\varepsilon'$  is sufficiently small positive real number. Thus

$$\begin{aligned}
&O(n^{3\eta-\frac{7}{2}-(\ell_1+\dots+\ell_k-1)/4} \exp(c_1\sqrt{an})) + O(n^{-(1-\eta)(\ell_1+\dots+\ell_k)-2\eta} \exp(c_1\sqrt{an})) \\
&= O(n^{-1+\varepsilon-(\ell_1+\dots+\ell_k)/4} \exp(c_1\sqrt{an})) \\
&= O(n^{d(\mathbf{l})-\frac{1}{4}+\varepsilon} \exp(c_1\sqrt{an})).
\end{aligned}$$

Noting that

$$\exp(c_1\sqrt{an}) = \exp\left(\pi \sqrt{\frac{2a(\mathbf{s}, \mathbf{l})n}{3}}\right),$$

we obtain a proof of Theorem 1.1.

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## References

- [1] Z. Ahmed, N. Baruah, M.G. Dastidar, *New congruences modulo 5 for the number of 2-color partitions*, J. Number Theory 157 (2015), 184–98, <http://dx.doi.org/10.1016/j.jnt.2015.05.002>.
- [2] R. Balasubramanian, F. Luca, *On the number of factorizations of an integer*, Integers 11 (2011), 139–143, <http://dx.doi.org/10.1515/integ.2011.012>, A12, 5 pp.
- [3] H.C. Chan, *Ramanujan’s cubic continued fraction and an analog of his “most beautiful identity”*, Int. J. Number Theory 6 (2010), no. 3, 673–680, <http://dx.doi.org/10.1142/S1793042110003150>.
- [4] H.C. Chan, S. Cooper, *Congruences modulo powers of 2 for a certain partition function*, Ramanujan J. 22 (2010), 101–117. <http://dx.doi.org/10.1007/s11139-009-9197-6>.
- [5] S.C. Chen, *Congruences for a certain partition function*, Ann. Comb. 18 (2014), 607–615, <http://dx.doi.org/10.1007/s00026-014-0240-y>.
- [6] S. Chern, *New congruences for 2-color partitions*, J. Number Theory 163 (2016), 474–481, <http://dx.doi.org/10.1016/j.jnt.2015.12.020>.
- [7] Y.-G. Chen, Y.-L. Li, *On the square-root partition function*, C. R. Math. Acad. Sci. Paris 353 (4) (2015), 287–290, <http://dx.doi.org/10.1016/j.crma.2015.01.013>.
- [8] P. Erdős, *On an elementary proof of some asymptotic formulas in the theory of partitions*, Ann. Math. 43 (1942), 437–450, <http://dx.doi.org/10.2307/1968802>.

- [9] G.H. Hardy, S. Ramanujan, *Asymptotic formula for the distribution of integers of various types*, Proc. London Math. Soc. (2) 16 (1917), 112–132.
- [10] A.E. Ingham, *A Tauberian theorem for partitions*, Ann. of Math. (2) 42 (1941), 1075–1090, <http://dx.doi.org/10.2307/1970462>.
- [11] B. Kim, *An analog of crank for a certain kind of partition function arising from the cubic continued fraction*, Acta Arith. 148 (2011), no. 1, 1–19, <http://dx.doi.org/10.4064/aa148-1-1>.
- [12] D.H. Lehmer, *On the remainders and convergence of the series for the partition function*, Trans. Amer. Math. Soc. 46 (1939), 362–373, <http://dx.doi.org/10.2307/1989927>.
- [13] Y. L. Li and Y. G. Chen, *On the  $r$ -th Root Partition Function*, Taiwanese J. Math. 20 (2016), 545–551, <http://dx.doi.org/10.11650/tjm.20.2016.6812>.
- [14] F. Luca and D. Ralaivaosaona, *An explicit bound for the number of partitions into roots*, J. Number Theory 169 (2016), 250–264, <http://dx.doi.org/10.1016/j.jnt.2016.05.017>.
- [15] A.M. Odlyzko, *Asymptotic enumeration methods*, Handbook of combinatorics. 1, 2 (1995), 1063–1229.
- [16] S. Ramanujan, *Congruence properties of partitions*, Math. Z. 9 (1921), no. 1-2, 147–153, <http://dx.doi.org/10.1007/BF01378341>.
- [17] J. Sinick, *Ramanujan congruences for a class of eta quotients*, Int. J. Number Theory 6 (2010), 835–847, <http://dx.doi.org/10.1142/S1793042110003253>.
- [18] Y.V. Uspensky *Asymptotic expressions of numerical functions occurring in problems concerning the partition of numbers into summands*, Bull Acad Sci de Russie 14 (1920), 199–218.